Wang *Fixed Point Theory and Applications* 2013, **2013**:96
http://www.fixedpointtheoryandapplications.com/content/2013/1/96

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Coincidence point theorems for *G*-isotone mappings in partially ordered metric spaces

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Abstract
We establish coincidence and common fixed-point theorems for *G*-isotone mappings in partially ordered metric spaces, which include the corresponding coupled, tripled and quadruple fixed-point theorems as special cases. Our proofs are simpler and essentially different from the ones devoted to coupled, tripled and quadruple fixed-point problems that appeared in the last years.

MSC: 47H10; 54H25

Keywords: *G*-isotone mapping; coincidence point; common fixed point; partially ordered metric space

1 Introduction

The Banach contraction principle is the most celebrated fixed-point theorem. There are great number of generalizations of the Banach contraction principle. A very recent trend in metrical fixed-point theory, initiated by Ran and Reurings [1], and continued by Nieto and Lopez [2, 3], Bhaskar and Lakshmikantham [4] and many other authors, is to consider a partial order on the ambient metric space (X,d) and to transfer a part of the contractive property of the nonlinear operators into its monotonicity properties. This approach turned out to be very productive; see, for example, [1–9], and the obtained results found important applications to the existence of solutions for matrix equations or ordinary differential equations and integral equations, see [1–4, 9] and reference therein.

In 2006, Bhaskar and Lakshmikantham [4] introduced the notion of *coupled fixed point* and proved some fixed-point theorems under certain conditions. Later, Lakshmikantham and Ćirić [8] extended these results by defining the *mixed g-monotone property*, *coupled coincidence point* and *coupled common fixed point*. On the other hand, Berinde and Borcut [6] introduced the concept of *tripled fixed point* and proved some related theorems. Later, Borcut and Berinde [10] extended these results by defining the *mixed g-monotone property*, *tripled coincidence point* and *tripled common fixed point*. These results were then extended and generalized by several authors in the last five years; see [5, 6, 11–23] and reference therein. Recently, Karapınar [24] introduced the notion of *quadruple fixed* point and proved some related fixed-point theorems in partially ordered metric space (see also [24–27]). Berzig and Samet [28] extended and generalized the mentioned fixed-point results to higher dimensions. However, they used permutations of variables and distinguished between the first and the last variables. Very recently, Roldan *et al.* [29] extend the mentioned previous results for non-linear mappings of any number of arguments, not necessarily permuted or ordered, in the framework of partially ordered complete metric spaces. We remind the reader of the following fact: in order to guarantee the existence of coupled (tripled or quadruple) coincidence point, the authors constructed two (three or four) Cauchy sequences using the properties of mixed monotone mappings and contractive conditions. It is not easy to prove that two (three or four) sequences are simultaneous Cauchy sequences. Then we spontaneously wonder the following questions:

Question 1.1 Can we obtain more general fixed-point theorems including the corresponding coupled, tripled and quadruple fixed-point theorems as three special cases?

Question 1.2 Can we provide a new method for approximating coupled, tripled and quadruple fixed points?

In this work, motivated and inspired by the above results, we establish more general fixed-point theorems including the coupled, tripled and quadruple fixed-point theorems as three special cases. Furthermore, we provide affirmative answers to Questions 1.1 and 1.2. The main results extend and improve the recent corresponding results in the literature. Our works bring at least two new features to coupled, tripled and quadruple fixed-point theory. First, we provide a new method for approximating coupled, tripled and quadruple fixed points. Second, our proofs are simpler and essentially different from the ones devoted to coupled, tripled and quadruple fixed-point problems that appeared in the last years.

2 Preliminaries

For simplicity, we denote from now on $\underbrace{X \times X \cdots X \times X}_{k}$ by X^k, where $k \in \mathbb{N}$ and X is a non-empty set. Let n be a positive integer, $\varphi^n(t)$ will denote the function $\varphi^n(t) = \underbrace{\varphi \circ \varphi \cdots \varphi \circ \varphi}_{n}(t)$. If elements x, y of a partially ordered set $(X, \leq)$ are comparable (*i.e.* $x \leq y$ or $y \leq x$ holds) we will write $x \asymp y$.

Let Φ denote the set of all functions $\varphi : [0,\infty) \to [0,\infty)$, which satisfies

(i_φ) $\varphi(t) < t$ for all $t \in (0,\infty)$;
(ii_φ) $\lim_{r \to t^+} \varphi(r) < t$ for all $t \in (0,\infty)$.

Definition 2.1 [12] Let $(R, \leq)$ be a partially ordered set and d be a metric on R. We say that $(R, d, \leq)$ is regular if the following conditions hold:

(i) if a non-decreasing sequence $\{x_n\}$ is such that $x_n \to x$, then $x_n \leq x$ for all n,
(ii) if a non-increasing sequence $\{y_n\}$ is such that $y_n \to y$, then $y_n \geq y$ for all n.

Definition 2.2 [8] Let $(X, \leq)$ be a partially ordered set and $F : X^2 \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if F is monotone g-non-decreasing in its first argument and is monotone g-non-increasing in its second argument, that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \quad g(x_1) \leq g(x_2) \quad \text{implies} \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad g(y_1) \le g(y_2) \quad \text{implies} \quad F(x, y_1) \ge F(x, y_2).$$

Definition 2.3 [10] Let $(X, \le)$ be a partially ordered set and two mappings $F : X^3 \to X$, $g : X \to X$. We say that F has the mixed g-monotone property if $F(x, y, z)$ is g-monotone non-decreasing in x, it is g-monotone non-increasing in y and it is g-monotone non-decreasing in z, that is, for any $x, y, z \in X$,

$$x_1, x_2 \in X, \quad g(x_1) \le g(x_2) \quad \Rightarrow \quad F(x_1, y, z) \le F(x_2, y, z),$$
$$y_1, y_2 \in X, \quad g(y_1) \le g(y_2) \quad \Rightarrow \quad F(x, y_1, z) \ge F(x, y_2, z)$$

and

$$z_1, z_2 \in X, \quad g(z_1) \le g(z_2) \quad \Rightarrow \quad F(x, y, z_1) \le F(x, y, z_2).$$

Note that if g is the identity mapping, then Definitions 2.2 and 2.3 reduce to Definition 1.1 in [4] and Definition 4 in [6] of mixed monotone property, respectively.

Definition 2.4 [29] Let $F : X^4 \to X$ be a mapping. We say that F has the mixed monotone property if $F(x, y, z, w)$ is monotone non-decreasing in x and z, and it is monotone non-increasing in y and w, that is, for any $x, y, z, w \in X$

$$x_1, x_2 \in X, \quad x_1 \le x_2 \quad \Rightarrow \quad F(x_1, y, z, w) \le F(x_2, y, z, w),$$
$$y_1, y_2 \in X, \quad y_1 \le y_2 \quad \Rightarrow \quad F(x, y_1, z, w) \ge F(x, y_2, z, w),$$
$$z_1, z_2 \in X, \quad z_1 \le z_2 \quad \Rightarrow \quad F(x, y, z_1, w) \le F(x, y, z_2, w),$$
$$w_1, w_2 \in X, \quad w_1 \le w_2 \quad \Rightarrow \quad F(x, y, z, w_1) \ge F(x, y, z, w_2).$$

Some authors introduced the concept of *coincidence point* in different ways and with different names. Let $F : X^k \to X$ and $g : X \to X$ be two mappings.

Definition 2.5 A point $(x_1, x_2, \dots, x_k) \in X^k$ is:
- (i) a coupled coincidence point [8] if $k = 2$, $F(x_1, x_2) = g(x_1)$ and $F(x_2, x_1) = g(x_2)$,
- (ii) a tripled coincidence point [10] if $k = 3$, $F(x_1, x_2, x_3) = g(x_1)$, $F(x_2, x_1, x_2) = g(x_2)$ and $F(x_3, x_2, x_1) = g(x_3)$,
- (iii) a coupled common fixed point [8] if $k = 2$, $F(x_1, x_2) = g(x_1) = x_1$ and $F(x_2, x_1) = g(x_2) = x_2$,
- (iv) a tripled common fixed point [10] if $k = 3$, $F(x_1, x_2, x_3) = g(x_1) = x_1$, $F(x_2, x_1, x_2) = g(x_2) = x_2$ and $F(x_3, x_2, x_1) = g(x_3) = x_3$,
- (v) a quadruple fixed point [24] if $k = 4$, $F(x, y, z, w) = x$, $F(y, z, w, x) = y$, $F(z, w, x, y) = z$ and $F(w, x, y, z) = w$,
- (vi) a Φ-coincidence point [29] if $k = n$, $F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \dots, x_{\sigma_i(n)}) = gx_{\tau_i}$ for all i.

Similarly, note that if g is the identity mapping, then *coupled coincidence point*, *tripled coincidence point* and Φ-*coincidence point* reduce to *coupled fixed point* (Gnana-Bhaskar and Lakshmikantham [4]), *tripled fixed point* (Berinde and Borcut [6]) and Φ-*fixed point* [29], respectively.

Definition 2.6 We say that the mappings $F : X^k \to X$ and $g : X \to X$ are commutative
 (i) if $k = 2$, $g(F(x_1, x_2)) = F(g(x_1), g(x_2))$ for all $x_1, x_2 \in X$ [8],
 (ii) if $k = 3$, $g(F(x_1, x_2, x_3)) = F(g(x_1), g(x_2), g(x_3))$ for all $x_1, x_2, x_3 \in X$ [10].

Let $(X, \leq)$ be a partially ordered set and d be a metric on X. We endow the product space X^k with the following partial order: for $(y^1, y^2, \dots, y^i, \dots, y^k), (v^1, v^2, \dots, v^i, \dots, v^k) \in X^k$,

$$\bigl(y^1, y^2, \dots, y^i, \dots, y^k\bigr) \leq \bigl(v^1, v^2, \dots, v^i, \dots, v^k\bigr) \quad \Leftrightarrow \quad \begin{cases} y^i \leq v^i, & \text{if } i = 1, 3, 5, \dots, \\ y^i \geq v^i, & \text{if } i = 2, 4, 6, \dots, \end{cases} \tag{2.1}$$

which will be denoted in the sequel, for convenience, by $\leq$, also. Obviously, $(X^k, \leq)$ is a partially ordered set. The mapping $\rho_k : X^k \times X^k \to [0, +\infty)$, given by

$$\rho_k(Y, V) = \frac{1}{k}\bigl[d\bigl(y^1, v^1\bigr) + d\bigl(y^2, v^2\bigr) + \cdots + d\bigl(y^k, v^k\bigr)\bigr], \tag{2.2}$$

where $Y = (y^1, y^2, \dots, y^k)$, $V = (v^1, v^2, \dots, v^k) \in X^k$, defines a metric on X^k. It is easy to see that

$$Y_n \to Y \quad (n \to \infty) \quad \Leftrightarrow \quad y_n^i \to y^i \quad (n \to \infty), i = 1, 2, \dots, k, \tag{2.3}$$

where $Y_n = (y_n^1, y_n^2, \dots, y_n^k)$, $Y = (y^1, y^2, \dots, y^k) \in X^k$. Indeed, $[Y_n \to Y \ (n \to \infty) \Leftrightarrow y_n^i \to y^i \ (n \to \infty)$ for all $i] \iff [\rho_k(Y_n, Y) \to 0 \ (n \to \infty) \Leftrightarrow d(y_n^i, y_i) \to 0 \ (n \to \infty)$ for all $i]$.

In order to prove our main results, we need the following lemma.

Lemma 2.7 *Let $(X, \leq)$ be a partially ordered set and d be a metric on X. If $(X, \leq, d)$ is regular, then $(X^k, \leq, \rho_k)$ is regular.*

Proof Without loss of generality, we assume that the sequence $\{Y_n\}$ is non-decreasing with $Y_n \to Y$ $(n \to \infty)$, where $Y_n = (y_n^1, y_n^2, \dots, y_n^k)$, $Y = (y^1, y^2, \dots, y^k) \in X^k$. From (2.3), we have

$$y_n^i \to y^i \quad (n \to \infty), i = 1, 2, \dots, k. \tag{2.4}$$

Now suppose that $i = 1, 3, 5, \dots$. As the sequence $\{Y_n\}$ is non-decreasing and (2.1), we have the sequences $\{y_n^i\}_{n=1}^{\infty}$ are non-decreasing. From (2.4), the regularity of $(X, \leq, d)$ and using Definition 2.1, we have

$$y_n^i \leq y^i, \quad i = 1, 3, 5, \dots. \tag{2.5}$$

Suppose that $i = 2, 4, 6, \dots$. Since the sequence $\{Y_n\}$ is non-decreasing and (2.1), we have the sequences $\{y_n^i\}_{n=1}^{\infty}$ are non-increasing. From (2.4), the regularity of $(X, \leq, d)$ and using Definition 2.1, we have

$$y_n^i \geq y^i, \quad i = 2, 4, 6, \dots. \tag{2.6}$$

By (2.1), (2.5) and (2.6), we have $Y_n \leq Y$ for all n. By analogy, we show that if a non-increasing sequence $\{Y_n\}$ is such that $Y_n \to Y$, then $Y_n \geq Y$ for all n. Therefore, $(X^k, \leq, \rho_k)$ is regular. □

3 Main results

We now state and prove the main results of this paper.

Definition 3.1 We say that the mappings $T : X^k \to X^k$ and $G : X^k \to X^k$ are commutative if $TG(Y) = GT(Y)$ for all $Y \in X^k$.

Definition 3.2 Let $(X^k, \leq)$ be a partially ordered set and $T : X^k \to X^k$, $G : X^k \to X^k$. We say that T is a G-isotone mapping if, for any $Y_1, Y_2 \in X^k$

$$G(Y_1) \leq G(Y_2) \quad \Rightarrow \quad T(Y_1) \leq T(Y_2).$$

Definition 3.3 An element $Y \in X^k$ is called a coincidence point of the mappings $T : X^k \to X^k$ and $G : X^k \to X^k$ if $T(Y) = G(Y)$. Furthermore, if $T(Y) = G(Y) = Y$, then we say that Y is a common fixed point of T and G.

Theorem 3.4 *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $G : X^k \to X^k$ and $T : X^k \to X^k$ be a G-isotone mapping for which there exists $\varphi \in \Phi$ such that for all $Y \in X^k$, $V \in X^k$ with $G(Y) \geq G(V)$,*

$$\rho_k\big(T(Y), T(V)\big) \leq \varphi\big(\rho_k\big(G(Y), G(V)\big)\big), \tag{3.1}$$

where ρ_k is defined via (2.2). *Suppose $T(X^k) \subset G(X^k)$ and also suppose either*

(a) *T is continuous, G is continuous and commutes with T or*

(b) *$(X, d, \leq)$ is regular and $G(X^k)$ is closed.*

If there exists $Y_0 \in X^k$ such that $G(Y_0) \asymp T(Y_0)$, then T and G have a coincidence point.

Proof Since $T(X^k) \subset G(X^k)$, it follows that there exists $Y_1 \in X^k$ such that $G(Y_1) = T(Y_0)$. In general, there exists $Y_n \in X^k$ such that $G(Y_{n+1}) = T(Y_n)$, $n \geq 0$. We denote $Z_0 = G(Y_0)$ and

$$Z_{n+1} = G(Y_{n+1}) = T(Y_n), \quad n \geq 0. \tag{3.2}$$

Obviously, if $Z_{n+1} = Z_n$ for some $n \geq 0$, then there is nothing to prove. So, we may assume that $Z_{n+1} \neq Z_n$ for all $n \geq 0$. Since $G(Y_0) \asymp T(Y_0)$, without loss of generality, we assume that $G(Y_0) \leq T(Y_0)$ (the case $G(Y_0) \geq T(Y_0)$ is similar), that is, $Z_0 \leq Z_1$. Assume that $Z_{n-1} \leq Z_n$, that is, $G(Y_{n-1}) \leq G(Y_n)$. Since T is a G-isotone mapping, we get

$$Z_n = T(Y_{n-1}) \leq T(Y_n) = Z_{n+1},$$

which shows that $Z_n \leq Z_{n+1}$ for all $n \geq 0$. This actually means that the sequence $\{Z_n\}_{n=0}^{\infty}$ is non-decreasing. Since $G(Y_n) = Z_n \geq G(Y_{n-1}) = Z_{n-1}$, from (3.1) and (i_φ) we have

$$\begin{aligned}
\rho_k(Z_{n+1}, Z_n) &= \rho_k\big(T(Y_n), T(Y_{n-1})\big) \\
&\leq \varphi\big(\rho_k\big(G(Y_n), G(Y_{n-1})\big)\big) \\
&= \varphi\big(\rho_k(Z_n, Z_{n-1})\big) \\
&< \rho_k(Z_n, Z_{n-1})
\end{aligned} \tag{3.3}$$

for all $n \geq 1$. Hence, the sequence $\{\delta_n\}_{n=0}^{\infty}$ given by $\delta_n = \rho_k(Z_{n+1}, Z_n)$ is monotone decreasing and bounded below. Therefore, there exists some $\delta \geq 0$ such that $\lim_{n\to\infty} \delta_n = \delta$. We shall prove that $\delta = 0$. Assume that $\delta > 0$. Then by letting $n \to \infty$ in (3.3) and (ii_φ) we have

$$\delta \leq \lim_{n\to\infty} \varphi(\delta_{n-1}) = \lim_{r\to\delta^+} \varphi(r) < \delta,$$

which is a contradiction. Thus,

$$\lim_{n\to\infty} \delta_n = 0. \tag{3.4}$$

We claim that $\{Z_n\}$ is a Cauchy sequence. Indeed, if it is false, then there exist $\epsilon > 0$ and the sequences $\{Z_{m(t)}\}$ and $\{Z_{n(t)}\}$ of $\{Z_n\}$ such that $n(t)$ is the minimal in the sense that $n(t) > m(t) \geq t$ and $\rho_k(Z_{m(t)}, Z_{n(t)}) > \epsilon$. Therefore, $\rho_k(Z_{m(t)}, Z_{n(t)-1}) \leq \epsilon$.

Using the triangle inequality, we obtain

$$\begin{aligned}\epsilon &< \rho_k(Z_{m(t)}, Z_{n(t)}) \leq \rho_k(Z_{m(t)}, Z_{n(t)-1}) + \rho_k(Z_{n(t)-1}, Z_{n(t)}) \\ &\leq \epsilon + \rho_k(Z_{n(t)-1}, Z_{n(t)}).\end{aligned}$$

Letting $t \to \infty$ in the above inequality and using (3.4), we get

$$\lim_{t\to\infty} \rho_k(Z_{m(t)}, Z_{n(t)}) = \epsilon^+. \tag{3.5}$$

Since $n(t) > m(t)$, we have $Z_{m(t)} \leq Z_{n(t),}$ and hence $G(Y_{n(t)}) \geq G(Y_{m(t)})$. Now, by (3.1), we have

$$\begin{aligned}\rho_k(Z_{n(t)+1}, Z_{m(t)+1}) &= \rho_k\bigl(T(Y_{n(t)}), T(Y_{m(t)})\bigr) \\ &\leq \varphi\bigl(\rho_k\bigl(G(Y_{n(t)}), G(Y_{m(t)})\bigr)\bigr) = \varphi\bigl(\rho_k(Z_{n(t)}, Z_{m(t)})\bigr).\end{aligned}$$

Observe that

$$\begin{aligned}\rho_k(Z_{m(t)}, Z_{n(t)}) &\leq \rho_k(Z_{m(t)}, Z_{m(t)+1}) + \rho_k(Z_{m(t)+1}, Z_{n(t)+1}) + \rho_k(Z_{n(t)+1}, Z_{n(t)}) \\ &\leq \delta_{m(t)} + \delta_{n(t)} + \rho_k(Z_{m(t)+1}, Z_{n(t)+1}) \\ &\leq \delta_{m(t)} + \delta_{n(t)} + \varphi\bigl(\rho_k(Z_{n(t)}, Z_{m(t)})\bigr).\end{aligned}$$

Letting $t \to \infty$ in the above inequality and using (3.4)-(3.5), we have

$$\epsilon \leq \lim_{t\to\infty} \varphi(r_t) = \lim_{r\to\epsilon^+} \varphi(r) < \epsilon,$$

where $r_t = \rho_k(Z_{n(t)}, Z_{m(t)})$, which is a contradiction. Hence, the sequence $\{Z_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the metric space (X^k, ρ_k). On the other hand, since (X, d) is a complete metric space, thus the metric space (X^k, ρ_k) is complete. Therefore, there exists $\bar{Z} \in X^k$ such that $\lim_{n\to\infty} Z_n = \bar{Z}$, that is, $\lim_{n\to\infty} G(Y_n) = \bar{Z}$.

Now suppose that the assumption (a) holds. By the continuity of G, we have $\lim_{n\to\infty} G(G(Y_{n+1})) = G(\bar{Z})$. On the other hand, by the commutativity of T and G, we have

$$G\bigl(G(Y_{n+1})\bigr) = G\bigl(T(Y_n)\bigr) = T\bigl(G(Y_n)\bigr). \tag{3.6}$$

By (3.6) and the continuity of T, we have

$$G(\bar{Z}) = \lim_{n\to\infty} G\big(G(Y_{n+1})\big) = \lim_{n\to\infty} T\big(G(Y_n)\big) = T(\bar{Z}),$$

which shows that $\bar{Z}$ is a coincidence point of T and G.

Suppose that the assumption (b) holds. Using Lemma 2.7, we have $(X^k, \leq, \rho_k)$ is regular. Since $\{Z_n\}_{n=0}^{\infty}$ is non-decreasing sequence that converges to $\bar{Z}$, in view of Definition 2.1, we have $Z_n \leq \bar{Z}$ for all n. Since $G(X^k)$ is closed and by (3.2), we obtain that there exists $\bar{Y} \in X^k$ for which

$$\lim_{n\to\infty} G(Y_n) = \lim_{n\to\infty} T(Y_n) = \bar{Z} = G(\bar{Y}).$$

Then from (3.1), we have

$$\rho_k\big(T(Y_n), T(\bar{Y})\big) \leq \varphi\big(\rho_k\big(G(Y_n), G(\bar{Y})\big)\big)$$

for all $n \geq 0$. Letting $n \to \infty$ in the above inequality, we have $\rho_k(G(\bar{Y}), T(\bar{Y})) = 0$, which implies that $G(\bar{Y}) = T(\bar{Y})$. Therefore, $\bar{Y}$ is a coincidence point of T and G. □

Remark 3.5 Different kinds of contractive conditions are studied and we use a distinct methodology to prove Theorem 3.4. The authors proved that any number of sequences are simultaneous Cauchy sequence in [29]. However, we only need to proof that one sequence is a Cauchy sequence.

Taking $k = 1$ in Theorem 3.4, we can obtain the following result immediately.

Corollary 3.6 *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $G : X \to X$ and $T : X \to X$ be a G-isotone mapping for which there exists $\varphi \in \Phi$ such that for all $Y \in X$, $V \in X$ with $G(Y) \geq G(V)$,*

$$d\big(T(Y), T(V)\big) \leq \varphi\big(d\big(G(Y), G(V)\big)\big).$$

Suppose $T(X) \subset G(X)$ and also suppose either

(a) *T is continuous, G is continuous and commutes with T or*
(b) *$(X, d, \leq)$ is regular and $G(X)$ is closed.*

If there exists $Y_0 \in X$ such that $G(Y_0) \asymp T(Y_0)$, then T and G have a coincidence point.

Now, we will show that Theorem 3.4 allow us to derive coupled, tripled and quadruple fixed-point theorems for mixed monotone mappings in partially ordered metric space.

Taking $k = 2$, $T(Y) = (F(x, y), F(y, x))$ and $G(Y) = (g(x), g(y))$ for $Y = (x, y) \in X^2$ in Theorem 3.4, we can obtain the following result.

Corollary 3.7 *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $g : X \to X$ and $F : X^2 \to X$ be a mixed g-monotone mapping for which there exists $\varphi \in \Phi$ such that for all $x, y, u, v \in X$ with $g(x) \geq g(u)$, $g(y) \leq g(v)$,*

$$d\big(F(x, y), F(u, v)\big) + d\big(F(y, x), F(v, u)\big) \leq 2\varphi\left(\frac{d(g(x), g(u)) + d(g(y), g(v))}{2}\right). \tag{3.7}$$

Suppose $F(X^2) \subset g(X)$ *and also suppose either*

(a) F *is continuous*, g *is continuous and commutes with* F *or*

(b) $(X, d, \leq)$ *is regular and* $g(X)$ *is closed.*

If there exist $x_0, y_0 \in X$ *such that*

$$g(x_0) \leq F(x_0, y_0) \quad \text{and} \quad g(y_0) \geq F(y_0, x_0), \tag{3.8}$$

or

$$g(x_0) \geq F(x_0, y_0) \quad \text{and} \quad g(y_0) \leq F(y_0, x_0), \tag{3.9}$$

then there exist $\bar{x}, \bar{y} \in X$ *such that* $g(\bar{x}) = F(\bar{x}, \bar{y})$ *and* $g(\bar{y}) = F(\bar{y}, \bar{x})$, *that is*, F *and* g *have a couple coincidence point.*

Proof For simplicity, we denote $Y = (x, y)$, $V = (u, v)$ and $Y_n = (x_n, y_n)$ for all $n \geq 0$. We endow the product space X^2 with the following partial order:

$$\text{for } Y, V \in X^2, \quad Y \leq V \quad \Leftrightarrow \quad x \leq u, \qquad y \geq v. \tag{3.10}$$

Consider the function $\rho_2 : X^2 \times X^2 \to [0, +\infty)$ defined by

$$\rho_2(Y, V) = \frac{1}{2}\big[d(x, u) + d(y, v)\big], \quad \forall Y, V \in X^2. \tag{3.11}$$

Obviously, $(X^2, \leq)$ and ρ_2 are two particular cases of $(X^k, \leq)$ and ρ_k defined by (2.1) and (2.2), respectively. Now consider the operators $T : X^2 \to X^2$ and $G : X^2 \to X^2$ defined by

$$T(Y) = \big(F(x, y), F(y, x)\big) \tag{3.12}$$

and

$$G(Y) = \big(g(x), g(y)\big), \quad \forall Y \in X^2. \tag{3.13}$$

Since $F(X^2) \subset g(X)$, we have $T(X^2) \subset G(X^2)$.

We claim that T is a G-isotone mapping. Indeed, suppose that $G(Y_1) \leq G(Y_2)$, Y_1, $Y_2 \in X^2$. By (3.10) and (3.13), we have $g(x_1) \leq g(x_2)$ and $g(y_1) \geq g(y_2)$. Since F is g-mixed monotone, we have

$$F(x_1, y_1) \leq F(x_2, y_2) \quad \text{and} \quad F(y_1, x_1) \geq F(y_2, x_2), \quad \forall G(Y_1) \leq G(Y_2). \tag{3.14}$$

From (3.10), (3.12) and (3.14), we have

$$T(Y_1) = \big(F(x_1, y_1), F(y_1, x_1)\big) \leq \big(F(x_2, y_2), F(y_2, x_2)\big) = T(Y_2), \quad \forall G(Y_1) \leq G(Y_2).$$

Similarly, we can obtain that for any $Y_1, Y_2 \in X^2$, $G(Y_1) \geq G(Y_2) \Rightarrow T(Y_1) \geq T(Y_2)$. By (3.8)-(3.10), we have there exists $Y_0 \in X^2$ such that $G(Y_0) \asymp T(Y_0)$.

From (3.11) and (3.12), we have

$$\frac{d(F(x, y), F(u, v)) + d(F(y, x), F(v, u))}{2} = \rho_2\big(T(Y), T(V)\big)$$

and

$$\varphi\left(\frac{d(g(x),g(u))+d(g(y),g(v))}{2}\right)=\varphi\big(\rho_2\big(G(Y),G(V)\big)\big)$$

for any $Y\in X^2$, $V\in X^2$. It follows from (3.7) that

$$\rho_2\big(T(Y),T(V)\big)\le\varphi\big(\rho_2\big(G(Y),G(V)\big)\big),\quad\forall G(Y)\ge G(V).$$

Now suppose that the assumption (a) holds. By the continuity of g, we have G is continuous. From (3.12), (3.13) and using the commutativity of F and g, we have, for any $Y\in X^2$

$$\begin{aligned}TG(Y)&=T\big(g(x),g(y)\big)=\big(F\big(g(x),g(y)\big),F\big(g(y),g(x)\big)\big)\\&=\big(g\big(F(x,y)\big),g\big(F(y,x)\big)\big)=GT(Y),\end{aligned}$$

which implies that G commutes with T. It is easy to see that T is continuous. Indeed, by (3.11), we obtain that $Y_n\to Y$ $(n\to\infty)$ if and only if $x_n\to x$ and $y_n\to y$ $(n\to\infty)$. Since F is continuous, we have $F(x_n,y_n)\to F(x,y)$ and $F(y_n,x_n)\to F(y,x)$ $(n\to\infty)$, for any $Y_n\to Y$ $(n\to\infty)$. Therefore, we have

$$T(Y_n)=\big(F(x_n,y_n),F(y_n,x_n)\big)\to\big(F(x,y),F(y,x)\big)=T(Y)\quad(n\to\infty)$$

for any $Y_n\to Y$ $(n\to\infty)$.

Suppose that the assumption (b) holds. It is easy to see that $G(X^2)$ is closed.

All the hypothesis of Theorem 3.4 ($k=2$) are satisfied, and so we deduce the existence of a coincidence point of T and G. From (3.12) and (3.13), there exists $(\bar{x},\bar{y})$ such that $g(\bar{x})=F(\bar{x},\bar{y})$ and $g(\bar{y})=F(\bar{y},\bar{x})$, that is, $(\bar{x},\bar{y})$ is a coupled coincidence point of F and g. □

Remark 3.8 Note that in the case of the condition (b) satisfied in Corollary 3.7, we omit the control conditions: g is continuous and commutes with F, which are needed in the proof of Theorem 2.1 in [8] and Theorem 3 in [7].

Taking $k=3$, $T(Y)=(F(x,y,z),F(y,x,y),F(z,y,x))$ and $G(Y)=(g(x),g(y),g(z))$ for $Y=(x,y,z)\in X^3$ in Theorem 3.4, we can obtain the following result by the similar argument as we did in the proof of Corollary 3.7.

Corollary 3.9 *Let $(X,\le)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F:X^3\to X$ and $g:X\to X$ such that F has the mixed g-monotone property and $F(X^3)\subset g(X)$. Assume there is a function $\varphi\in\Phi$ such that*

$$\begin{aligned}&d\big(F(x,y,z),F(u,v,w)\big)+d\big(F(y,x,y),F(v,u,v)\big)+d\big(F(z,y,x),F(w,v,u)\big)\\&\quad\le3\varphi\left(\frac{d(g(x),g(u))+d(g(y),g(v))+d(g(z),g(w))}{3}\right)\end{aligned}$$

for any $x,y,z,u,v,w\in X$ for which $g(x)\ge g(u)$, $g(v)\ge g(y)$ and $g(z)\ge g(w)$. Suppose either

(a) *F is continuous, g is continuous and commutes with F or*
(b) *$(X, \leq, d)$ is regular and $g(X)$ is closed.*
If there exist $x_0, y_0, z_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0, z_0), \qquad g(y_0) \geq F(y_0, x_0, y_0) \quad \text{and} \quad g(z_0) \leq F(z_0, y_0, x_0),$$

or

$$g(x_0) \geq F(x_0, y_0, z_0), \qquad g(y_0) \leq F(y_0, x_0, y_0) \quad \text{and} \quad g(z_0) \geq F(z_0, y_0, x_0),$$

then there exist $x, y, z \in X$ such that

$$F(x, y, z) = g(x), \qquad F(y, x, y) = g(y) \quad \text{and} \quad F(z, y, x) = g(z),$$

that is, F and g have a tripled coincidence point.

Similarly, taking $k = 4$, $T(Y) = (F(x, y, z, w), F(y, z, w, x), F(z, w, x, y), F(w, x, y, z))$ and G is the identity mapping on X^4 for $Y = (x, y, z, w) \in X^4$ in Theorem 3.4, we can obtain the following result.

Corollary 3.10 *Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X^4 \to X$ such that F has the mixed monotone property. Assume there is a function $\varphi \in \Phi$ such that*

$$\begin{aligned} &d\bigl(F(x, y, z, w), F(u, v, r, t)\bigr) + d\bigl(F(y, z, w, x), F(v, r, t, u)\bigr) + d\bigl(F(z, w, x, y), F(r, t, u, v)\bigr) \\ &\quad + d\bigl(F(w, x, y, z), F(t, u, v, r)\bigr) \leq 4\varphi\left(\frac{d(x, u) + d(y, v) + d(z, r) + d(w, t)}{4}\right) \end{aligned}$$

for any $x, y, z, w, u, v, r, t \in X$ for which $x \geq u$, $y \leq v$, $z \geq r$ and $w \leq t$. Suppose either
(a) *F is continuous or*
(b) *$(X, \leq, d)$ is regular.*
If there exist $x_0, y_0, z_0, w_0 \in X$ such that

$$\begin{aligned} &x_0 \leq F(x_0, y_0, z_0, w_0), \qquad y_0 \geq F(y_0, z_0, w_0, x_0), \\ &z_0 \leq F(z_0, w_0, x_0, y_0) \quad \text{and} \quad w_0 \geq F(w_0, x_0, y_0, z_0), \end{aligned}$$

or

$$\begin{aligned} &x_0 \geq F(x_0, y_0, z_0, w_0), \qquad y_0 \leq F(y_0, z_0, w_0, x_0), \\ &z_0 \geq F(z_0, w_0, x_0, y_0) \quad \text{and} \quad w_0 \leq F(w_0, x_0, y_0, z_0), \end{aligned}$$

then there exist $x, y, z, w \in X$ such that

$$F(x, y, z, w) = x, \qquad F(y, z, w, x) = y, \qquad F(z, w, x, y) = z \quad \text{and} \quad F(w, x, y, z) = w,$$

that is, F have a quadruple fixed point.

Theorem 3.11 *In addition to the hypothesis of Theorem* 3.4, *suppose that for every* $\bar{Y}, Y^* \in X^k$ *there exists* $V \in X^k$ *such that* $T(V)$ *is comparable to* $T(\bar{Y})$ *and to* $T(Y^*)$. *Also, assume that* φ *is non-decreasing. Let* G *commute with* T *if the assumption* (b) *holds. Then* T *and* G *have a unique common fixed point, that is, there exists a unique point* $\bar{Z} \in X^k$ *such that* $\bar{Z} = G(\bar{Z}) = T(\bar{Z})$.

Proof From Theorem 3.4, the set of coincidence points of T and G is non-empty. Assume that $\bar{Y}$ and $Y^* \in X^k$ are two coincidence points of T and G. We shall prove that $G(\bar{Y}) = G(Y^*)$. Put $V_0 = V$ and choose $V_1 \in X^k$ so that $G(V_1) = T(V_0)$. Then, similarly to the proof of Theorem 3.4, we obtain the sequence $\{G(V_n)\}_{n=1}^{\infty}$ defined as follows: $G(V_{n+1}) = T(V_n)$, $n \geq 0$. Since $T(\bar{Y}) = G(\bar{Y})$ and $T(V) = G(V_1)$ are comparable, without loss of generality, we assume that $G(\bar{Y}) \leq G(V_1)$. Since T is a G-isotone mapping, we have

$$G(\bar{Y}) = T(\bar{Y}) \leq T(V_1) = G(V_2).$$

Recursively, we get that $G(\bar{Y}) \leq G(V_n)$, $\forall n \geq 1$. Thus, by the contractive condition (3.1), one gets

$$\rho_k\big(G(V_{n+1}), G(\bar{Y})\big) = \rho_k\big(T(V_n), T(\bar{Y})\big) \leq \varphi\big(\rho_k\big(G(V_n), G(\bar{Y})\big)\big).$$

Thus, by the above inequality, we get

$$\Delta_{n+1} \leq \varphi(\Delta_n), \quad n \geq 0,$$

where $\Delta_n = \rho_k(G(V_n), G(\bar{Y}))$. Since φ is non-decreasing, it follows that

$$\Delta_{n+1} \leq \varphi^n(\Delta_1).$$

From the definition of Φ, we get $\lim_{n\to\infty} \varphi^n(t) = 0$, for each $t > 0$. Then, we have $\lim_{n\to\infty} \Delta_n = 0$. Thus,

$$\lim_{n\to\infty} \rho_k\big(G(\bar{Y}), G(V_n)\big) = 0. \tag{3.15}$$

Similarly, we obtain that

$$\lim_{n\to\infty} \rho_k\big(G\big(Y^*\big), G(V_n)\big) = 0. \tag{3.16}$$

Combining (3.15) and (3.16) yields that $G(\bar{Y}) = G(Y^*)$. Since $G(\bar{Y}) = T(\bar{Y})$, by the commutativity of T and G, we have

$$G\big(G(\bar{Y})\big) = G\big(T(\bar{Y})\big) = T\big(G(\bar{Y})\big). \tag{3.17}$$

Denote $G(\bar{Y}) = \bar{Z}$. By (3.17), we have $G(\bar{Z}) = T(\bar{Z})$, that is $\bar{Z}$ is a coincidence point of T and G. Thus, we have $G(\bar{Z}) = G(\bar{Y}) = \bar{Z}$. Therefore, $\bar{Z}$ is a common fixed point of T and G.

To prove the uniqueness, assume Z^* is another common fixed point of T and G. Then we have

$$Z^* = G\big(Z^*\big) = G(\bar{Z}) = \bar{Z}. \qquad \square$$

Corollary 3.12 *In addition to the hypothesis of Corollary* 3.7, *suppose that for every* $(\bar{x},\bar{y}),(x^*,y^*)\in X^2$ *there exists* $(u,v)\in X\times X$ *such that* $(F(u,v),F(v,u))$ *is comparable to* $(F(x^*,y^*),F(y^*,x^*))$ *and to* $(F(\bar{x},\bar{y}),F(\bar{y},\bar{x}))$. *Also, assume that* φ *is non-decreasing. Let* g *commute with* F *if the assumption* (b) *holds. Then* F *and* g *have a unique coupled common fixed point, that is, there exists a unique point* $(\bar{z},\bar{w})\in X^2$ *such that*

$$\bar{z}=g(\bar{z})=F(\bar{z},\bar{w})\quad\textit{and}\quad\bar{w}=g(\bar{w})=F(\bar{w},\bar{z}).$$

Proof Similarly to the proof of Corollary 3.7, we can obtain all conditions of Theorem 3.4 ($k=2$) are satisfied. In addition, by the commutativity of g and F, we have G commutes with T. For simplicity, we denote $\bar{Y}=(\bar{x},\bar{y})$, $Y^*=(x^*,y^*)$ and $V=(u,v)\in X^2$. By (3.12), we have

$$T(V)=\big(F(u,v),F(v,u)\big),\qquad T(\bar{Y})=\big(F(\bar{x},\bar{y}),F(\bar{y},\bar{x})\big)\quad\text{and}$$
$$T\big(Y^*\big)=\big(F\big(x^*,y^*\big),F\big(y^*,x^*\big)\big).$$

By hypothesis, there exists $V\in X^2$ such that $T(V)$ is comparable to $T(\bar{Y})$ and to $T(Y^*)$. Hence, there is no doubt that all conditions of Theorem 3.11 are satisfied ($k=2$). Therefore, there exists a unique point $\bar{Z}=(\bar{z},\bar{w})\in X^2$ such that $\bar{Z}=G(\bar{Z})=T(\bar{Z})$. That is, $\bar{z}=g(\bar{z})=F(\bar{z},\bar{w})$ and $\bar{w}=g(\bar{w})=F(\bar{w},\bar{z})$. □

By the similar argument as we did in the proof of Corollary 3.12, we deduce the following corollary from Theorem 3.11 ($k=3$).

Corollary 3.13 *In addition to the hypothesis of Corollary* 3.9, *suppose that for all* (x,y,z) *and* (u,v,r) *in* X^3, *there exists* (a,b,c) *in* X^3 *such that* $(F(a,b,c),F(b,a,b),F(c,b,a))$ *is comparable to* $(F(x,y,z),F(y,x,y),F(z,y,x))$ *and* $(F(u,v,r),F(v,u,v),F(r,v,u))$. *Also, assume that* φ *is non-decreasing. Let* g *commute with* F *if the assumption* (b) *holds. Then* F *and* g *have a unique tripled common fixed point* (x,y,z), *that is*,

$$x=g(x)=F(x,y,z),\qquad y=g(y)=F(y,x,y)\quad\textit{and}\quad z=g(z)=F(z,y,x).$$

Competing interests
The author declares that she has no competing interests.

Author's contributions
SW completed the paper herself. The author read and approved the final manuscript.

Acknowledgements
The author thanks the editor and the referees for their useful comments and suggestions. This study was supported by the Natural Science Foundation of Yancheng Teachers University under Grant (12YCKL001) and UNSF of Jiangsu province, China (09KJD110005).

Received: 31 July 2012 Accepted: 26 March 2013 Published: 12 April 2013

References

1. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**(5), 1435-1443 (2004)
2. Nieto, JJ, Rodriguez-Lopez, R: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**(3), 223-239 (2005)
3. Nieto, JJ, Rodriguez-Lopez, R: Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations Acta Math. Sin. Engl. Ser. **23**(12), 2205-2212 (2007)

4. Bhaskar, TG, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. **65**(7), 1379-1393 (2006)
5. Agarwal, RP, El-Gebeily, MA, O'Regan, D: Generalized contractions in partially ordered metric spaces. Appl. Anal. **87**, 1-8 (2008)
6. Berinde, V, Borcut, M: Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces. Nonlinear Anal. **74**, 4889-4897 (2011)
7. Berinde, V: Coupled coincidence point theorems for mixed point monotone nonlinear operators. Comput. Math. Appl. **64**, 1770-1777 (2012)
8. Lakshmikantham, V, Ćirić, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70**, 4341-4349 (2009)
9. Luong, NV, Thuan, NX: Coupled fixed points in partially ordered metric spaces and application. Nonlinear Anal. **74**, 983-992 (2011)
10. Borcut, M, Berinde, V: Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. Appl. Math. Comput. **218**(10), 5929-5936 (2012)
11. Abbas, M, Khan, MA, Radenović, S: Common coupled fixed point theorems in cone metric space for ω-compatible mappings. Appl. Math. Comput. **217**, 195-203 (2010)
12. Aydi, H, Karapınar, E, Postolache, M: Tripled coincidence theorems for weak φ-contractions in partially ordered metric spaces. Fixed Point Theory Appl. **2012**, 44 (2012). doi:10.1186/1687-1812-2012-44
13. Aydi, H, Karapınar, E: Tripled fixed points in ordered metric spaces. Bull. Math. Anal. Appl. **4**(1), 197-207 (2012)
14. Aydi, H, Karapınar, E, Radenovic, S: Tripled coincidence fixed point results for Boyd-Wong and Matkowski type contractions. Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. (2012). doi:10.1007/s13398-012-0077-3
15. Aydi, H, Vetro, C, Karapınar, E: Meir-Keeler type contractions for tripled fixed points. Acta Math. Sci. **32**(6), 2119-2130 (2012)
16. Aydi, H, Karapınar, E: New Meir-Keeler type tripled fixed point theorems on ordered partial metric spaces. Math. Probl. Eng. **2012**, Article ID 409872 (2012)
17. Aydi, H, Karapınar, E, Shatnawi, W: Tripled fixed point results in generalized metric spaces. J. Appl. Math. **2012**, Article ID 314279 (2012)
18. Borcut, M: Tripled coincidence theorems for contractive type mappings in partially ordered metric spaces. Appl. Math. Comput. **218**, 7339-7346 (2012)
19. Choudhury, BS, Kundu, A: A coupled coincidence point result in partially ordered metric spaces for compatible mappings. Nonlinear Anal. **73**, 2524-2531 (2010)
20. Samet, B: Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric space. Nonlinear Anal. **74**(12), 4508-4517 (2010)
21. Shatanawi, W: Partially ordered cone metric spaces and couple fixed point results. Comput. Math. Appl. **60**, 2508-2515 (2010)
22. Roldan, A, Martinez-Moreno, J, Roldan, C: Tripled fixed point theorem in fuzzy metric spaces and applications. Fixed Point Theory Appl. **2013**, 29 (2013). doi:10.1186/1687-1812-2013-29
23. Karapınar, E: Coupled fixed point theorems for nonlinear contractions in cone metric spaces. Comput. Math. Appl. **59**(12), 3656-3668 (2010)
24. Karapınar, E: Quartet fixed point for nonlinear contraction. http://arxiv.org/abs/1106.5472
25. Karapınar, E: Quadruple fixed point theorems for weak ϕ-contractions. ISRN Math. Anal. **2011**, Article ID 989423 (2011)
26. Karapınar, E, Berinde, V: Quadruple fixed point theorems for nonlinear contractions in partially ordered metric spaces. Banach J. Math. Anal. **6**, 74-89 (2012)
27. Karapınar, E: A new quartet fixed point theorem for nonlinear contractions. JP J. Fixed Point Theory Appl. **6**, 119-135 (2011)
28. Berzig, M, Samet, B: An extension of coupled fixed point's concept in higher dimension and applications. Comput. Math. Appl. **63**(8), 1319-1334 (2012)
29. Roldan, A, Martinez-Moreno, J, Roldan, C: Multidimensional fixed point theorems in partially ordered metric spaces. J. Math. Anal. Appl. **396**, 536-545 (2012)

doi:10.1186/1687-1812-2013-96
Cite this article as: Wang: **Coincidence point theorems for *G*-isotone mappings in partially ordered metric spaces.** *Fixed Point Theory and Applications* 2013 **2013**:96.